On examples of intermediate subfactors from conformal field theory

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Abstract

Motivated by our subfactor generalization of Wall's conjecture, in this paper we determine all intermediate subfactors for conformal subnets corresponding to four infinite series of conformal inclusions, and as a consequence we verify that these series of subfactors verify our conjecture. Our results can be stated in the framework of Vertex Operator Algebras. We also verify our conjecture for Jones-Wassermann subfactors from representations of Loop groups extending our earlier results.

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1 Introduction

Let M be a factor represented on a Hilbert space and N a subfactor of M which is irreducible, i.e.,$N' \cap M = \mathbb{C}$. Let K be an intermediate von Neumann subalgebra for the inclusion $N \subset M$. Note that $K' \cap K \subset N' \cap M = \mathbb{C}$, K is automatically a factor. Hence the set of all intermediate subfactors for $N \subset M$ forms a lattice under two natural operations $\wedge$ and $\vee$ defined by:

$$K_1 \wedge K_2 = K_1 \cap K_2, K_1 \vee K_2 = (K_1 \cup K_2)''.$$

The commutant map $K \to K'$ maps an intermediate subfactor $N \subset K \subset M$ to $M' \subset K' \subset N'$. This map exchanges the two natural operations defined above.

Let $M \subset M_1$ be the Jones basic construction of $N \subset M$. Then $M \subset M_1$ is canonically isomorphic to $M' \subset N'$, and the lattice of intermediate subfactors for $N \subset M$ is related to the lattice of intermediate subfactors for $M \subset M_1$ by the commutant map defined as above.

Let G_1 be a group and G_2 be a subgroup of G_1. An interval sublattice $[G_1/G_2]$ is the lattice formed by all intermediate subgroups $K, G_2 \subseteq K \subseteq G_1$.

By cross product construction and Galois correspondence, every interval sublattice of finite groups can be realized as intermediate subfactor lattice of finite index. Hence the study of intermediate subfactor lattice of finite index is a natural generalization of the study of interval sublattice of finite groups. The study of intermediate subfactors has been very active in recent years(cf. [4],[12], [27], [25], [22], and [39] for a partial list).

In 1961 G. E. Wall conjectured that the number of maximal subgroups of a finite group G is less than $|G|$, the order of G (cf. [31]). In the same paper he proved his conjecture when G is solvable. See [28] for more recent result on Wall's conjecture.

Wall's conjecture can be naturally generalized to a conjecture about maximal elements in the lattice of intermediate subfactors. What we mean by maximal elements are those subfactors $K \neq M, N$ with the property that if K_1 is an intermediate subfactor and $K \subset K_1$, then $K_1 = M$ or K. Minimal elements are defined similarly where N is not considered as an minimal element. When M is the cross product of N by a finite group G, the maximal elements correspond to maximal subgroups of G, and the order of G is the dimension of second higher relative commutant. Hence a natural generalization of Wall's conjecture as proposed in [37] is the following:

Conjecture 1.1. *Let $N \subset M$ be an irreducible subfactor with finite index. Then the number of maximal intermediate subfactors is less than dimension of $N' \cap M_1$ (the dimension of second higher relative commutant of $N \subset M$).*

We note that since maximal intermediate subfactors in $N \subset M$ correspond to minimal intermediate subfactors in $M \subset M_1$, and the dimension of second higher relative commutant remains the same, the conjecture is equivalent to a similar conjecture as above with maximal replaced by minimal.

In [37],[14], Conjecture 1.1 is verified for subfactors coming from certain conformal field theories and subfactors which are more closely related to groups and more

generally Hopf algebras. In this paper we investigate Conjecture 1.1 for conformal subnets $\mathcal{A} \subset \mathcal{B}$ (cf. Definition 2.1) with finite index. Then Conjecture 1.1 in this case states:

Conjecture 1.2. *Suppose that conformal subnets $\mathcal{A} \subset \mathcal{B}$ (cf. Definition 2.1) has finite index. Then the number of minimal (resp. maximal) subnets between $\mathcal{A}$ and $\mathcal{B}$ is less than the dimension of the space of bounded maps from the vacuum representation of $\mathcal{B}$ to itself which commutes with the action of $\mathcal{A}$.*

In the above conjecture we have included both maximal and minimal cases since the dual of conformal subnet $\mathcal{A} \subset \mathcal{B}$ is not conformal subnet. It is also straightforward to phrase the above conjecture in terms of Vertex Operator Algebras (VOAs) and its sub-VOAs.

Note that any finite group G is embedded in a finite symmetric group S_n, and using the theory of permutation orbifolds as in [38] we can always find a completely rational net $\mathcal{B}$ such that G acts properly on $\mathcal{B}$ and with fixed point subnet $\mathcal{A}$. In this case the intermediate subnets between $\mathcal{A}$ and $\mathcal{B}$ are in one to one correspondence with subgroups of G. So in this orbifold case the minimal version of Conjecture 1.2 is equivalent to Wall's conjecture. Hence Conjecture 1.2 is highly nontrivial even if we assume that $\mathcal{B}$ is completely rational.

Though the orbifold case of Conjecture 1.2 in general is out of reach at present, there are very interesting other examples of subnets coming from conformal field theory (CFT). A large class of such examples come from conformal inclusions (cf. §2.5), and they provide a large class of subfactors which are not related to groups. In view of Conjecture 1.2 it is a natural question to investigate intermediate subnets of such examples, and this is the main goal of our paper.

Our results Th. 3.8, Th. 3.11 give a complete list of intermediate conformal subnets in subnets coming from four infinite series of maximal conformal inclusions, and as consequence, we are able to verify Conjecture 1.2 in these examples. Our results show that the intermediate subnets in these examples are very rare. The key idea behind the proof of Th. 3.8 is the property of induced adjoint representation: Prop. 3.6 shows that such induced representation is always irreducible when the intermediate subnet does not have additional weight 1 element. By locality consideration in Lemma 2.8 this forces the intermediate subnet to be simply simple current extensions when it has no additional weight 1 element. In the case when the intermediate net has additional weight 1 element, we use smeared vertex operators as in [34] and maximality of conformal inclusions to show that the intermediate subnet is in fact the largest net. The proof makes use of the analogue of statement in VOA theory that weight 1 element of a VOA forms a Lie algebra. The proof of Th. 3.11 is much simpler and make use of normal inclusions as in §4.2 of [35].

By using properties of smeared vertex operators in §3.1, we can translate our results in Th. 3.8, Th. 3.11 into statements about intermediate VOAs (cf. Th. 3.14). We think it is an interesting question to find a VOA proof of Th. 3.14.

In §4 we extend our earlier results in §5 of [39] on Jones-Wassermann subfactors and we verify that these subfactors verify Conjecture 1.1.

In addition to what are already described as above, we have included a preliminary section §2 where we introduce the basic notion of conformal nets, subnets , conformal inclusions, and induction to describe the background of our results in §3 and §4.

2 Preliminaries

2.1 Preliminaries on sectors

Given an infinite factor M, the *sectors of M* are given by

$$\mathrm{Sect}(M) = \mathrm{End}(M)/\mathrm{Inn}(M),$$

namely $\mathrm{Sect}(M)$ is the quotient of the semigroup of the endomorphisms of M modulo the equivalence relation: $\rho, \rho' \in \mathrm{End}(M)$, $\rho \sim \rho'$ iff there is a unitary $u \in M$ such that $\rho'(x) = u\rho(x)u^*$ for all $x \in M$.

$\mathrm{Sect}(M)$ is a *-semiring (there are an addition, a product and an involution $\rho \to \bar{\rho}$) equivalent to the Connes correspondences (bimodules) on M up to unitary equivalence. If ρ is an element of $\mathrm{End}(M)$ we shall denote by $[\rho]$ its class in $\mathrm{Sect}(M)$. We define $\mathrm{Hom}(\rho, \rho')$ between the objects $\rho, \rho' \in \mathrm{End}(M)$ by

$$\mathrm{Hom}(\rho, \rho') \equiv \{a \in M : a\rho(x) = \rho'(x)a \ \forall x \in M\}.$$

We use $\langle \lambda, \mu \rangle$ to denote the dimension of $\mathrm{Hom}(\lambda, \mu)$; it can be ∞, but it is finite if λ, μ have finite index. See [26] for the definition of index for type II_1 case which initiated the subject and [29] for the definition of index in general. Also see §2.3 [18] for expositions. $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have if ν has finite index, then $\langle \nu\lambda, \mu \rangle = \langle \lambda, \bar{\nu}\mu \rangle$, $\langle \lambda\nu, \mu \rangle = \langle \lambda, \mu\bar{\nu} \rangle$ which follows from Frobenius duality. μ is a subsector of λ if there is an isometry $v \in M$ such that $\mu(x) = v^*\lambda(x)v, \forall x \in M$. We will also use the following notation: if μ is a subsector of λ, we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$. A sector is said to be irreducible if it has only one subsector.

2.2 Local nets

By an interval of the circle we mean an open connected non-empty subset I of S^1 such that the interior of its complement I' is not empty. We denote by $\mathcal{I}$ the family of all intervals of S^1.

A *net* $\mathcal{A}$ of von Neumann algebras on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

from $\mathcal{I}$ to von Neumann algebras on a fixed separable Hilbert space $\mathcal{H}$ that satisfies:

A. *Isotony.* If $I_1 \subset I_2$ belong to $\mathcal{I}$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

If $E \subset S^1$ is any region, we shall put $\mathcal{A}(E) \equiv \bigvee_{E \supset I \in \mathcal{I}} \mathcal{A}(I)$ with $\mathcal{A}(E) = \mathbb{C}$ if E has empty interior (the symbol $\vee$ denotes the von Neumann algebra generated).

The net $\mathcal{A}$ is called *local* if it satisfies:

B. *Locality.* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$ then

$$[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},$$

where brackets denote the commutator.

The net $\mathcal{A}$ is called *Möbius covariant* if in addition satisfies the following properties **C,D,E,F**:

C. *Möbius covariance.* There exists a non-trivial strongly continuous unitary representation U of the Möbius group $\mathsf{M\ddot{o}b}$ (isomorphic to $PSU(1,1)$) on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \; = \; \mathcal{A}(gI), \quad g \in \mathsf{M\ddot{o}b}, \; I \in \mathcal{I}.$$

D. *Positivity of the energy.* The generator of the one-parameter rotation subgroup of U (conformal Hamiltonian), denoted by L_0 in the following, is positive.

E. *Existence of the vacuum.* There exists a unit U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

By the Reeh-Schlieder theorem Ω is cyclic and separating for every fixed $\mathcal{A}(I)$. The modular objects associated with $(\mathcal{A}(I), \Omega)$ have a geometric meaning

$$\Delta_I^{it} = U(\Lambda_I(2\pi t)), \qquad J_I = U(r_I) \; .$$

Here Λ_I is a canonical one-parameter subgroup of $\mathsf{M\ddot{o}b}$ and $U(r_I)$ is a antiunitary acting geometrically on $\mathcal{A}$ as a reflection r_I on S^1.

This implies *Haag duality*:

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I} \; ,$$

where I' is the interior of $S^1 \setminus I$.

F. *Irreducibility.* $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible iff Ω is the unique U-invariant vector (up to scalar multiples). Also $\mathcal{A}$ is irreducible iff the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are either $\mathbb{C}$ or III_1-factors with separable predual in Connes classification of type III factors.

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

G. *Conformal covariance.* There exists a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of $\mathsf{M\ddot{o}b}$ such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(\phi)\mathcal{A}(I)U(\phi)^* \; &= \; \mathcal{A}(\phi . I), \quad \phi \in \mathrm{Diff}(S^1), \\ U(\phi)xU(\phi)^* \; &= \; x, \quad x \in \mathcal{A}(I), \; \phi \in \mathrm{Diff}(I'), \end{aligned}$$

where $\mathrm{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of S^1 and $\mathrm{Diff}(I)$ the subgroup of diffeomorphisms g such that $\phi(z) = z$ for all $z \in I'$.

A (DHR) representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$ that associates to each I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} .$$

π is said to be Möbius (resp. diffeomorphism) covariant if there is a projective unitary representation U_π of $\mathsf{M\ddot{o}b}$ (resp. $\mathrm{Diff}(S^1)$) on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in \mathsf{M\ddot{o}b}$ (resp. $g \in \mathrm{Diff}(S^1)$).

By definition the irreducible conformal net is in fact an irreducible representation of itself and we will call this representation the *vacuum representation*.

Let G be a simply connected compact Lie group. By Th. 3.2 of [7], the vacuum positive energy representation of the loop group LG (cf. [30]) at level k gives rise to an irreducible conformal net denoted by $\mathcal{A}_{G_k}$. By Th. 3.3 of [7], every irreducible positive energy representation of the loop group LG at level k gives rise to an irreducible covariant representation of $\mathcal{A}_{G_k}$.

Given an interval I and a representation π of $\mathcal{A}$, there is an *endomorphism of* $\mathcal{A}$ *localized in* I equivalent to π; namely ρ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to π, such that $\rho_{I'} = \mathrm{id} \restriction \mathcal{A}(I')$. We now define the statistics. Given the endomorphism ρ of $\mathcal{A}$ localized in $I \in \mathcal{I}$, choose an equivalent endomorphism ρ_0 localized in an interval $I_0 \in \mathcal{I}$ with $\bar{I}_0 \cap \bar{I} = \emptyset$ and let u be a local intertwiner in $\mathrm{Hom}(\rho, \rho_0)$, namely $u \in \mathrm{Hom}(\rho_{\tilde{I}}, \rho_{0,\tilde{I}})$ with I_0 following clockwise I inside $\tilde{I}$ which is an interval containing both I and I_0.

The *statistics operator* $\epsilon(\rho, \rho) := u^*\rho(u) = u^*\rho_{\tilde{I}}(u)$ belongs to $\mathrm{Hom}(\rho^2_{\tilde{I}}, \rho^2_{\tilde{I}})$. We will call $\epsilon(\rho, \rho)$ the positive or right braiding and $\tilde{\epsilon}(\rho, \rho) := \epsilon(\rho, \rho)^*$ the negative or left braiding.

Let $\mathcal{B}$ be a conformal net. By a *conformal subnet* (cf. [22]) we shall mean a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset \mathcal{B}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{A}(I)$ of $\mathcal{B}(I)$, which is isotonic

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2), I_1 \subset I_2,$$

and conformal covariant with respect to the representation U, namely

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(g.I)$$

for all $g \in \mathrm{Diff}(S^1)$ and $I \in \mathcal{I}$. Note that by Lemma 13 of [22] for each $I \in \mathcal{I}$ there exists a conditional expectation $E_I : \mathcal{B}(I) \to \mathcal{A}(I)$ such that E_I preserves the vector state given by the vacuum of $\mathcal{A}$.

Definition 2.1. *Let $\mathcal{A}$ be a conformal net. A conformal net $\mathcal{B}$ on a Hilbert space $\mathcal{H}$ is an extension of $\mathcal{A}$ or $\mathcal{A}$ is a subnet of $\mathcal{B}$ if there is a DHR representation π of $\mathcal{A}$ on $\mathcal{H}$ such that $\pi(\mathcal{A}) \subset \mathcal{B}$ is a conformal subnet. The extension is irreducible if $\pi(\mathcal{A}(I))' \cap \mathcal{B}(I) = \mathbb{C}$ for some (and hence all) interval I, and is of finite index if $\pi(\mathcal{A}(I)) \subset \mathcal{B}(I)$ has finite index for some (and hence all) interval I. The index will be called the index of the inclusion $\pi(\mathcal{A}) \subset \mathcal{B}$ and is denoted by $[\mathcal{B} : \mathcal{A}]$. If π as representation of $\mathcal{A}$ decomposes as $[\pi] = \sum_\lambda m_\lambda[\lambda]$ where m_λ are non-negative integers and λ are irreducible DHR representations of $\mathcal{A}$, we say that $[\pi] = \sum_\lambda m_\lambda[\lambda]$ is the spectrum of the extension. For simplicity we will write $\pi(\mathcal{A}) \subset \mathcal{B}$ simply as $\mathcal{A} \subset \mathcal{B}$.*

Lemma 2.2. *If $\mathcal{A} \subset \mathcal{B}$ is a conformal subnet with finite index, then $\mathcal{A} \subset \mathcal{B}$ is irreducible.*

Proof. This is proved in Cor. 3.6 of [2], without assumption of conformal covariance of $\mathcal{A}$ but under the additional assumption that $\mathcal{A}$ is strongly additive to ensure the equivalence of local and global intertwiners, but for conformal net $\mathcal{A}$ the equivalence of local and global intertwiners for finite index representations are proved in §2 of [13], thus the proof of Cor. 3.6 of [2] applies verbatim. ∎

Lemma 2.3. *Suppose that $\mathcal{A} \subset \mathcal{B}$ has finite index , and let $[\pi] = \sum_\lambda m_\lambda[\lambda]$ be as in Definition above. Fix an interval I and suppose that $\lambda, \bar{\lambda}$ is localized on I.*

(1) Let $K_\lambda := \{T \in B(I) | Ta = a\lambda(a)T, \forall a \in \mathcal{A}(I)\}$. Then K_λ is a vector space of dimension $m_\lambda \leq d_\lambda$. One can find isometries $T_{\lambda_i} \in K_\lambda, T_{\lambda_i} \in K_{\bar{\lambda}}, 1 \leq i \leq m_\lambda$ such that $T_{\lambda_i} a = \lambda(a) T_{\lambda_i}, \forall a \in \mathcal{A}, E(T_{\lambda_i} T_{\lambda_j^*}) = \delta_{ij} 1/d_\lambda, E(T_{\bar{\lambda}_i} T_{\bar{\lambda}_j^*}) = \delta_{ij} 1/d_\lambda, T_{\lambda_i}^* \in \mathcal{A}(I) T_{\bar{\lambda}_i}$; Every $b \in \mathcal{B}(I)$ can be written as $b = \sum_{\lambda_i} d_\lambda T_{\lambda_i}^* E(T_{\lambda_i} b)$;

(2) Let $L_\lambda \subset K_\lambda$ be subspaces with the following properties:(a) $L_\lambda L_\mu \subset \sum_\nu \mathcal{A}(I) L_\nu$; (b) $L_\lambda^* \subset \mathcal{A}(I) L_{\bar{\lambda}}$. Then there is an intermediate subnet $\mathcal{A} \subset \mathcal{C} \subset \mathcal{B}$ such that $\mathcal{C}(I) = \sum_\lambda \mathcal{A}(I) L_\lambda$. Conversely every intermediate subnet arises this way;

(3) If Ω is the vacuum vector of $\mathcal{B}$, and denote by $\overline{\mathcal{A}\Omega} = H_0, \overline{T_{\lambda_i}^* \mathcal{A}\Omega} = H_{\lambda_i}$, then as Hilbert space $H = \bigoplus_{\lambda_i, 1 \leq i \leq m_\lambda} H_{\lambda_i}$, and the map $\sqrt{d_\lambda} T_{\lambda_i}^* : H_0 \to H_{\lambda_i}$ is a unitary intertwiner between the action of $\lambda(\mathcal{A}(I))$ on H_0 and $\mathcal{A}(I)$ on H_{λ_i}.

Proof. (1) and(2) follow from §3 of [25] and §2 of [22]. For (3), only unitarity has to be checked. We have

$$\langle T_{\lambda_i}^* a_1 \Omega, T_{\lambda_i}^* a_2 \Omega \rangle = \langle a_2^* E(T_{\lambda_i} T_{\lambda_i}^*) a_1 \Omega, \Omega \rangle = 1/d_\lambda \langle a_1 \Omega, a_2 \Omega \rangle, \forall a_1, a_2 \in \mathcal{A}(I),$$

and the proof is complete. ∎

2.3 Induced endomorphisms

Suppose a conformal net $\mathcal{A}$ and a representation λ is given. Fix an open interval I of the circle and Let $M := \mathcal{A}(I)$ be a fixed type III_1 factor. Then λ give rises to an endomorphism still denoted by λ of M. Suppose $\{[\lambda]\}$ is a finite set of all equivalence classes of irreducible, covariant, finite-index representations of an irreducible local

conformal net $\mathcal{A}$. We will use $\Delta_{\mathcal{A}}$ to denote all finite index representations of net $\mathcal{A}$ and will use the same notation $\Delta_{\mathcal{A}}$ to denote the corresponding sectors of M.

We will denote the conjugate of $[\lambda]$ by $[\bar{\lambda}]$ and identity sector (corresponding to the vacuum representation) by $[1]$ if no confusion arises, and let $N_{\lambda\mu}^{\nu} = \langle [\lambda][\mu], [\nu] \rangle$. Here $\langle \mu, \nu \rangle$ denotes the dimension of the space of intertwiners from μ to ν (denoted by $\mathrm{Hom}(\mu, \nu)$). The univalence of λ and the statistical dimension of (cf. §2 of [13]) will be denoted by ω_λ and $d(\lambda)$ (or d_λ)) respectively. Suppose that $\rho \in \mathrm{End}(M)$ has the property that $\gamma = \rho\bar{\rho} \in \Delta_{\mathcal{A}}$. By §2.7 of [23], we can find two isometries $v_1 \in \mathrm{Hom}(\gamma, \gamma^2), w_1 \in \mathrm{Hom}(1, \gamma)$ such that $\bar{\rho}(M)$ and v_1 generate M and

$$
\begin{aligned}
v_1^* w_1 &= v_1^* \gamma(w_1) = d_\rho^{-1} \\
v_1 v_1 &= \gamma(v_1) v_1
\end{aligned}
$$

By Thm. 4.9 of [23], we shall say that ρ is *local* if

$$
v_1^* w_1 = v_1^* \gamma(w_1) = d_\rho^{-1} \tag{1}
$$

$$
v_1 v_1 = \gamma(v_1) v_1 \tag{2}
$$

$$
\bar{\rho}(\epsilon(\gamma, \gamma)) v_1 = v_1 \tag{3}
$$

Note that if ρ is local, then

$$
\omega_\mu = 1, \forall \mu \prec \rho\bar{\rho} \tag{4}
$$

For each (not necessarily irreducible) $\lambda \in \Delta_{\mathcal{A}}$, let $\varepsilon(\lambda, \gamma) : \lambda\gamma \to \gamma\lambda$ (resp. $\tilde{\varepsilon}(\lambda, \gamma)$), be the positive (resp. negative) braiding operator as defined in Section 1.4 of [33]. Denote by $\lambda_\varepsilon \in \mathrm{End}(M)$ which is defined by

$$
\begin{aligned}
\lambda_\varepsilon(x) : &= ad(\varepsilon(\lambda, \gamma))\lambda(x) = \varepsilon(\lambda, \gamma)\lambda(x)\varepsilon(\lambda, \gamma)^* \\
\lambda_{\tilde{\varepsilon}}(x) : &= ad(\tilde{\varepsilon}(\lambda, \gamma))\lambda(x) = \tilde{\varepsilon}(\lambda, \gamma)^*\lambda(x)\tilde{\varepsilon}(\lambda, \gamma)^*, \forall x \in M.
\end{aligned}
$$

By (1) of Theorem 3.1 of [33], $\lambda_\varepsilon \rho(M) \subset \rho(M), \lambda_{\tilde{\varepsilon}} \rho(M) \subset \rho(M)$, hence the following definition makes sense:

Definition 2.4. *If $\lambda \in \Delta_{\mathcal{A}}$ define two elements of* $\mathrm{End}(M)$ *by*

$$
a_\lambda^\rho(m) := \rho^{-1}(\lambda_\varepsilon \rho(m)), \ \tilde{a}_\lambda^\rho(m) := \rho^{-1}(\lambda_{\tilde{\varepsilon}} \rho(m)), \forall m \in M.
$$

a_λ^ρ (resp. $\tilde{a}_\lambda^\rho$) will be referred to as positive (resp. negative) induction of λ with respect to ρ.

Remark 2.5. *For simplicity we will use $a_\lambda, \tilde{a}_\lambda$ to denote $a_\lambda^\rho, \tilde{a}_\lambda^\rho$ when it is clear that inductions are with respect to the same ρ.*

The endomorphisms a_λ are called braided endomorphisms in [33] due to its braiding properties (cf. (2) of Corollary 3.4 in [33]), and enjoy an interesting set of properties (cf. Section 3 of [33]). We summarize a few properties from [33] which will be used in this paper: (cf. Th. 3.1 , Co. 3.2 and Th. 3.3 of [33]):

Proposition 2.6. *(1). The maps* $[\lambda] \to [a_\lambda], [\lambda] \to [\tilde{a}_\lambda]$ *are ring homomorphisms;*

(2) $a_\lambda \bar{\rho} = \tilde{a}_\lambda \bar{\rho} = \bar{\rho}\lambda$*;*

(3) When $\rho\bar{\rho}$ *is local,* $\langle a_\lambda, a_\mu \rangle = \langle \tilde{a}_\lambda, \tilde{a}_\mu \rangle = \langle a_\lambda \bar{\rho}, a_\mu \bar{\rho} \rangle = \langle \tilde{a}_\lambda \bar{\rho}, \tilde{a}_\mu \bar{\rho} \rangle$*;*

(4) (3) remains valid if a_λ, a_μ *(resp.* $\tilde{a}_\lambda, \tilde{a}_\mu$*) are replaced by their subsectors. In particular we have* $\langle a_\lambda, \sigma \rangle = \langle \lambda, \rho\sigma\bar{\rho} \rangle$ *if* $\sigma \prec a_\mu$*.*

The following is Porp. 2.24 of [39]:

Proposition 2.7. *Suppose that* $\rho\bar{\rho} \in \Delta$*. Then:*

(1) ρ *is local iff* $\langle 1, a_\mu \rangle = \langle \rho\bar{\rho}, \mu \rangle, \forall \mu \in \Delta_{\mathcal{A}}$*;*

(2)

$$\rho = \rho' \rho'' = \tilde{\rho}' \tilde{\rho}''$$

where $\rho', \rho'', \tilde{\rho}', \tilde{\rho}'' \in \mathrm{End}(M)$*, and* $\rho', \tilde{\rho}'$ *are local which verifies*

$$\langle \rho' \bar{\rho}', \mu \rangle = \langle 1, a_\mu \rangle = \langle 1, a_\mu^{\rho'} \rangle$$
$$\langle \tilde{\rho}' \overline{\tilde{\rho}'}, \mu \rangle = \langle 1, \tilde{a}_\mu \rangle = \langle 1, \tilde{a}_\mu^{\tilde{\rho}'} \rangle$$

$\forall \mu \in \Delta_{\mathcal{A}}$*. We refer to* ρ' *(resp.* ρ''*) as the left (resp.right) local support of* ρ*.*

The following Lemma is Prop. 3.23 of [2] (The proof was also implicitly contained in the proof of Lemma 3.2 of [33]):

Lemma 2.8. *If* $\rho\bar{\rho}$ *is local, then* $[a_\lambda] = [\tilde{a}_\lambda]$ *iff* $\varepsilon(\lambda, \rho\bar{\rho})\varepsilon(\rho\bar{\rho}, \lambda) = 1$ *iff* $\varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda) = 1, \forall \mu \in \rho\bar{\rho}$*.*

We shall make use of the following notation in §4:

Definition 2.9. *For* $\lambda, \mu \in \Delta_{\mathcal{A}}$*,* $Z^\rho_{\lambda\mu} := \langle a_\lambda, \tilde{a}_\mu \rangle$*.*

2.4 Jones-Wassermann subfactors from representation of Loop groups

Let $G = SU(n)$. We denote LG the group of smooth maps $f : S^1 \mapsto G$ under pointwise multiplication. The diffeomorphism group of the circle $\mathrm{Diff}S^1$ is naturally a subgroup of $\mathrm{Aut}(LG)$ with the action given by reparametrization. In particular the group of rotations $\mathrm{Rot}S^1 \simeq U(1)$ acts on LG. We will be interested in the projective unitary representation $\pi : LG \to U(H)$ that are both irreducible and have positive energy. This means that π should extend to $LG \rtimes \mathrm{Rot}\ S^1$ so that $H = \oplus_{n \geq 0} H(n)$, where the $H(n)$ are the eigenspace for the action of $\mathrm{Rot}S^1$, i.e., $r_\theta \xi = \exp(in\theta)$ for $\theta \in H(n)$ and $\dim\ H(n) < \infty$ with $H(0) \neq 0$. It follows from [30] that for fixed level k which is a positive integer, there are only finite number of such irreducible representations indexed by the finite set

$$P^k_{++} = \left\{ \lambda \in P \mid \lambda = \sum_{i=1,\cdots,n-1} \lambda_i \Lambda_i, \lambda_i \geq 0\,, \sum_{i=1,\cdots,n-1} \lambda_i \leq k \right\}$$

where P is the weight lattice of $SU(n)$ and Λ_i are the fundamental weights. We will write $\lambda = (\lambda_1, ..., \lambda_{n-1}), \lambda_0 = k - \sum_{1\leq i\leq n-1} \lambda_i$ and refer to $\lambda_0, ..., \lambda_{n-1}$ as components of λ.

We will use Λ_0 or simply 1 to denote the trivial representation of $SU(n)$. For $\lambda, \mu, \nu \in P^k_{++}$, define $N^\nu_{\lambda\mu} = \sum_{\delta\in P^k_{++}} S^{(\delta)}_\lambda S^{(\delta)}_\mu S^{(\delta *)}_\nu / S^{(\delta}_{\Lambda_0})$ where $S^{(\delta)}_\lambda$ is given by the Kac-Peterson formula:

$$S^{(\delta)}_\lambda = c \sum_{w\in S_n} \varepsilon_w \exp(iw(\delta) \cdot \lambda 2\pi/n)$$

where $\varepsilon_w = \det(w)$ and c is a normalization constant fixed by the requirement that $S^{(\delta)}_\mu$ is an orthonormal system. It is shown in [17] P. 288 that $N^\nu_{\lambda\mu}$ are non-negative integers. Moreover, define $Gr(C_k)$ to be the ring whose basis are elements of P^k_{++} with structure constants $N^\nu_{\lambda\mu}$. The natural involution $*$ on P^k_{++} is defined by $\lambda \mapsto \lambda^* =$ the conjugate of λ as representation of $SU(n)$. Note that $\lambda \to \frac{S_{\lambda\mu}}{S_{1\mu}}$ gives a representation of $Gr(C_k)$.

We shall also denote $S^{(\Lambda)}_{\Lambda_0}$ by $S^{(\Lambda)}_1$. Define $d_\lambda = \frac{S^{(\lambda)}_1}{S^{(\Lambda_0)}_1}$. We shall call $(S^{(\delta)}_\nu)$ the S-matrix of $LSU(n)$ at level k.

We shall encounter the $\mathbb{Z}_n$ group of automorphisms of this set of weights, generated by

$$\sigma : \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}) \to \sigma(\lambda) = (k - 1 - \lambda_1 - \cdots \lambda_{n-1}, \lambda_1, \cdots, \lambda_{n-2}).$$

Define $\mathrm{col}(\lambda) = \Sigma_i(\lambda_i - 1)i$. $\mathrm{col}(\lambda)$ will be referred to as the color of λ. The central element $\exp \frac{2\pi i}{n}$ of $SU(n)$ acts on representation of $SU(n)$ labeled by λ as $\exp(\frac{2\pi i \mathrm{col}(\lambda)}{n})$. The irreducible positive energy representations of $LSU(n)$ at level k give rise to an irreducible conformal net $\mathcal{A}$ (cf. [18]) and its covariant representations. We will use $\lambda = (\lambda_1, ...\lambda_{n-1})$ to denote irreducible representations of $\mathcal{A}$ and also the corresponding endomorphism of $M = \mathcal{A}(I)$.

All the sectors $[\lambda]$ with λ irreducible generate the fusion ring of $\mathcal{A}$.

For λ irreducible, the univalence ω_λ is given by an explicit formula (cf. 9.4 of [PS]). Let us first define $h_\lambda = \frac{c_2(\lambda)}{k+n}$ where $c_2(\lambda)$ is the value of Casimir operator on representation of $SU(n)$ labeled by dominant weight λ. h_λ is usually called the conformal dimension. Then we have: $\omega_\lambda = exp(2\pi i h_\lambda)$. The conformal dimension of $\lambda = (\lambda_1, ..., \lambda_{n-1})$ is given by

$$h_\lambda = \frac{1}{2n(k+n)} \sum_{1\leq i\leq n-1} i(n-i)\lambda_i^2 + \frac{1}{n(k+n)} \sum_{1\leq j\leq i\leq n-1} j(n-i)\lambda_j\lambda_i + \frac{1}{2(k+n)} \sum_{1\leq j\leq n-1} j(n-j)\lambda_j \tag{5}$$

The following result is proved in [32] (See Corollary 1 of Chapter V in [32]).

Theorem 2.10. *Each $\lambda \in P^{(k)}_{++}$ has finite index with index value d^2_λ. The fusion ring generated by all $\lambda \in P^{(k)}_{++}$ is isomorphic to $Gr(C_k)$.*

Remark 2.11. *The subfactors in the above theorem are called Jones-Wassermann subfactors after the authors who first studied them (cf. [15],[32]).*

Definition 2.12. $v := (1, 0, ..., 0), v_0 := (1, 0, ..., 0, 1), \omega^i = k\Lambda_i, 0 \leq i \leq n-1$. v *(resp. v_0) will be referred to as vector (resp. adjoint) representation.*

The following is observed in [11]:

Lemma 2.13. *Let $(0, ..., 0, 1, 0, ...0)$ be the i-th ($1 \leq i \leq n-1$) fundamental weight. Then $[(0, ..., 0, 1, 0, ...0)\lambda]$ are determined as follows: $\mu \prec (0, ..., 0, 1, 0, ...0)\lambda$ iff when the Young diagram of μ can be obtained from Young diagram of λ by adding i boxes on i different rows of λ, and such μ appears in $[(0, ..., 0, 1, 0, ...0)\lambda]$ only once.*

Lemma 2.14. *(1) If $[\lambda] \neq \omega^i$ for some $0 \leq i \leq n-1$, then $v_0 \prec \lambda\bar{\lambda}$;*
(2) If $\lambda_1\lambda_2$ is irreducible, then either λ_1 or $\lambda_2 = \omega^i$ for some $0 \leq i \leq n-1$;
(3) Suppose that λ has color $0 mod n$. Then $\lambda \prec v_0^m$ for some $m \in \mathbb{N}$.

Proof. (1), (2) is lemma 2.30 of [39]. By the lemma above $\lambda \prec v^l$ for some $l \in \mathbb{N}$, and since $\mathrm{col}(\lambda) = 0 \mod n$, we have $l = nl_1, l_1 \in \mathbb{N}$. Since $1 \prec v^n, 1 \prec \bar{v}^n$, we have $[v^n] \prec [v^n\bar{v}^n] = ([v_0] + [1])^n$, and (3) follows. ∎

2.5 Subnets from conformal inclusions

Let $G \subset H$ be inclusions of compact simply connected Lie groups. $LG \subset LH$ is called a conformal inclusion if the level 1 projective positive energy representations of LH decompose as a finite number of irreducible projective representations of LG. $LG \subset LH$ is called a maximal conformal inclusion if there is no proper subgroup G' of H containing G such that $LG \subset LG'$ is also a conformal inclusion. A list of maximal conformal inclusions can be found in [24].

Let H^0 be the vacuum representation of LH, i.e., the representation of LH associated with the trivial representation of H. Then H^0 decomposes as a direct sum of irreducible projective representation of LG at level K. K is called the Dynkin index of the conformal inclusion.

We shall write the conformal inclusion as $G_K \subset H_1$. Note that it follows from the definition that $\mathcal{A}_{H_1}$ is an extension of $\mathcal{A}_{G_K}$. We shall limit our consideration to the following conformal inclusions so we can use the results of [33]:

$$SU(n)_{n-2} \subset SU\left(\frac{n(n-1)}{2}\right)_1, \quad N \geq 4; \tag{6}$$

$$SU(n)_{n+2} \subset SU\left(\frac{n(n+1)}{2}\right)_1; \tag{7}$$

$$SU(n)_n \subset Spin(n^2-1)_1, \quad N \geq 2; \tag{8}$$

$$SU(n)_m \times SU(m)_n \subset SU(mn)_1. \tag{9}$$

Note that except equation (9), the above cover all the infinite series of maximal conformal inclusions of the form $SU(N) \subset H$ with H being a simple group.

3 Intermediate subnets in confonmal subnets associated with conformal inclusions

Let $\mathcal{A} \subset \mathcal{B}$ be conformal subnets associated with conformal inclusions in §2.5, i.e., $\mathcal{A} = \mathcal{A}_{G_k} \subset \mathcal{B} = \mathcal{A}_{H_1}$. Our goal in this section is to list all intermediate subnets $\mathcal{A} \subset \mathcal{C} \subset \mathcal{B}$.

The spectrum $[\pi] = \sum_\lambda m_\lambda \lambda$ of $\mathcal{A} \subset \mathcal{B}$ is given by [1] and [21]. One interesting feature is that all $m_\lambda = 1$. We write $H_\mathcal{B} = \oplus_\lambda H_\lambda$ with H_0 the vacuum representation of $\mathcal{A}$, and $H_\mathcal{B}$ (resp. $H_\mathcal{C}$) the vacuum representation space of $\mathcal{B}$ (resp. $H_\mathcal{C}$).

Fix an interval I and let $M = \mathcal{A}(I) \subset \mathcal{C}(I), \rho \in \mathrm{End}(M), \rho\bar{\rho} = H_\mathcal{C} \in \Delta_\mathcal{A}$ where we use $H_\mathcal{C}$ to denote the restriction of the vacuum representation of $\mathcal{C}$ to $\mathcal{A}$. For $\lambda \in \Delta_\mathcal{A}$, we will write $a_\lambda^\mathcal{C} := a_\lambda^\rho$.

3.1 Smeared Vertex Operators

Let g (resp. h) be the Lie algebra of G (resp. H). Choose a basis $e_\alpha, e_{-\alpha}, h_\alpha$ in $h_\mathbb{C} := h \otimes \mathbb{C}$ with α ranging over the set of roots as in §2.5 of [30]. Let $X_\alpha := i(e_\alpha + e_{-\alpha}), Y_\alpha := (e_\alpha - e_{-\alpha})$. Denote by $\hat{h}$ the affine Kac-Moody algebra (cf. P. 163 of [20]) associated to $h_\mathbb{C}$. Note $\hat{h} = h_\mathbb{C} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, where $\mathbb{C}c$ is the 1-dimensional center of $\hat{h}$. For $X \in h$, Define $X(n) := X \otimes t^n$, $X(z) := \sum_n X(n) z^{-n-1}$ as on Page 312 of [19].

Let π^0 be the vacuum representation of LH_1 on $H_\mathcal{B}$ with vacuum vector Ω. Let D be the generator of the action of the rotation group on $H_\mathcal{B}$. $H_\mathcal{B}^0$ will denote the finite linear sum of the eigenvectors of D. For $\xi \in H_\mathcal{B}$, we define $||x||_s = ||(1+D)^s x||, s \in \mathbb{R}$. $H^s := \{x \in H_\mathcal{B}^0 | \ ||x||_s < \infty\}$ and $H(\infty) = \cap_{s \in \mathbb{R}} H^s$. Note that when $s \geq 0$, H^s is a complete space under the norm $||.||_s$. Clearly $H_\mathcal{B}^0 \subset H(\infty)$. The elements of $H_\mathcal{B}^0$ (resp. $H(\infty)$) will be called *finite energy vectors* (resp. *smooth vectors*). The eigenvalue of D is sometimes referred to as energy or weight.

Let us recall a few elementary facts about vertex operators which will be used. See [8] or [16] for an introduction on vertex operator algebras. Define $End(H_\mathcal{B}^0)$ to be the space of all linear operators (not necessarily bounded) from $H_\mathcal{B}^0$ to $H_\mathcal{B}^0$ and set

$$End(H_\mathcal{B}^0)[[z, z^{-1}]] := \{\sum_{n \in \mathbb{Z}} v_n z^n | v_n \in End(H_\mathcal{B}^0)\}.$$

By the statement on P. 154 of [9] which follows from Th. 2.4.1 of [9] there exists a linear map

$$\psi \in H_\mathcal{B}^0 \to V(\xi, z) = \sum_{m \in \mathbb{Z}} \psi(m) z^{-m-1} \in End(H_\mathcal{B}^0)[[z, z^{-1}]]$$

with the following properties:

(1) $\psi(-1)\Omega = \psi$;

(2) If

$$\psi = X_{i_1}(-1) ... X_{i_t}(-1)\Omega,$$

then

$$V(\psi, z) =: X_{i_1}(z)...X_{i_t}(z):$$

where $:, :$ are normal ordered products (cf. (2.38), (2.39) of [5]).

$V(\psi, z)$ is called a *vertex operator* of ψ.

Let $f = \sum_m f(m)z^m$ be a smooth test function. Define

$$||f||_s = \sum_{n\in\mathbb{Z}} (1 + |m|)^s |f(m)|.$$

The *smeared vertex operator* $V(\psi, f)$ is defined to be:

$$V(\psi, f) = \frac{1}{2\pi i}\int_{S^1} V(\psi, z) f dz = \sum_m f(m)\psi(m).$$

$V(\psi, f)$ is a well defined operator on $H^0_{\mathcal{B}}$. Let $V(\psi, f)^{FA}$ be the formal adjoint of $V(\psi, f)$ on $H^0_{\mathcal{B}}$. It is defined by the equation

$$\langle V(\psi, f)x, y\rangle = \langle x, V(\psi, f)^{FA} y\rangle, \forall x, y \in H^0_{\mathcal{B}}$$

where $\langle,\rangle$ is the inner product on Hilbert space $H^0_{\mathcal{B}}$.

Lemma 3.1. *The subspace spanned by $V(\psi, f)\Omega, \forall\psi \in H^0_\lambda = H_\lambda \cap H^0_{\mathcal{B}}, \forall f$ smooth, $suppf \in I$, is dense in H_λ.*

Proof. The proof is essentially the same as the proof of Reeh-Schlieder Th. Let $\xi \in H_\lambda$ be a vector which is orthogonal to the subspace spanned by $V(\psi, f)\Omega, \forall\psi \in H^0_\lambda, suppf \in I$. Suppose J is an open interval such that $\bar{J} \subset I$, and f is a smooth function with support in J. Consider the function

$$F(z) = \langle \exp(izD)\xi, V(\psi, f)\Omega\rangle.$$

Since the spectrum of D on H_λ is a subset of non-negative integers, it follows that $F(z)$ is holomorphic on the upper half plane, continues on the real line, and vanishes on an open interval on the real line. It follows by Schwartz reflection principle that $F(z)$ is identically zero, and we have

$$\langle \exp(itD)\xi, V(\psi, f)\Omega\rangle = \langle \xi, \exp(-itD)V(\psi, f)\exp(itD)\Omega\rangle = 0, \forall t \in \mathbb{R}.$$

On the other hand $\exp(-itD)V(\psi, f)\exp(itD)\Omega = V(\psi, R_t(f))\Omega$, where $R_t(f)(z) = f(\exp(it)z)$. Choose a covering of S^1 by intervals $R_{t_i}I, 1 \le n \le n$ and smooth functions f_i with support in $R_{t_i}I, 1 \le n \le n$ such that $\sum_{1\le i\le n} f_i = \frac{1}{z}$, then

$$0 = \langle \xi, \sum_{1\le i\le n} V(\psi, f_i)\Omega\rangle = \langle \xi, \psi\rangle, \forall\psi \in H^0_\lambda.$$

Since H^0_λ is dense in H_λ, we conclude that $\xi = 0$ and the lemma is proved. ∎

Recall $H_{\mathcal{B}} = \oplus_\lambda H_\alpha$ as representations of LG_k or $\mathcal{A}_{G_k}$. The lowest energy space of H_α, denoted by $H_\lambda(0)$ is a highest weight module of G with weight λ. The vertex operator

$$V(\psi, z) : H_\lambda(0) \to \mathrm{End}(H^0_{\mathcal{B}})[[z, z^{-1}]]$$

is a primary vertex operator for $\hat{g}$ with highest weight λ(cf. [19] and [9]). By a slightly abuse of notations we write such operator as $V(\lambda) = \sum_m V(\lambda)_m z^{-m-1}$.

Definition 3.2. *We define $V(\lambda)V(\mu)H_0$ to be the linear span of $V(\lambda)_m V(\mu)_n H^0_0, \forall n, m$.*

Note that by definition $V(\lambda)V(\mu)H_0$ is a $\hat{g}$ submodule of $H_{\mathcal{B}}$, and if $V(\lambda)V(\mu)H_0 \supset H^0_0$, then $\lambda = \bar{\mu}$.

The weight 1 element in $H_{\mathcal{B}}$ is a Lie algebra isomorphic to $h_{\mathbb{C}}$ and will be identified with $h_{\mathbb{C}}$. It has a subspace isomorphic to $g_{\mathbb{C}}$. The vertex operator associated with $h, V(h, z)$ is usually written as $h(z)$, similarly we write $V(h, f)$ as $h(f)$. $h(f)$ are skew adjoint unbounded operators if $f = f^*$. The orthogonal complement of $g_{\mathbb{C}}$ in $h_{\mathbb{C}}$, denoted by $h_{\mathbb{C}} \ominus g_{\mathbb{C}}$ is a direct sum of $H_\lambda(0)$ with $h_\lambda = 1$.

Lemma 3.3. *(1) Let T_λ, T_μ be as in Lemma 2.3. Then*

$$T^*_\lambda T^*_\mu H_0 \subset \overline{V(\lambda)V(\mu)H_0};$$

(2) If $E(T_\nu T^*_\lambda T^*_\mu) \neq 0$, then $H_\nu \subset \overline{V(\lambda)V(\mu)H_0}$;

Proof. Ad (1): We choose interval I_1 which is disjoint from I. By Lemma 3.1 it is sufficient to check that for all smooth f with support in I_1 and $\psi \in H^0_\mu$,

$$T^*_\lambda V(\psi, f)\Omega \in \overline{V(\lambda)V(\mu)H_0}.$$

By choosing H trivial in Prop. 2.3 of [34], we know that $V(\psi, f)$ is affiliated with $\mathcal{B}(I_1)$, and by locality we have $T^*_\lambda V(\psi, f)\Omega = V(\psi, f)T^*_\lambda\Omega$. Since $V(\mu)V(\lambda)H_0$ is a $\hat{g}$ module, the orthogonal complement of $V(\mu)V(\lambda)H_0$ is a direct of irreducible $\hat{g}$ module.

Now suppose that $\xi \in H^0_{\mathcal{B}}$ is orthogonal to $V(\mu)V(\lambda)H_0$. Choose $\chi_n \in H^0_\lambda$ such that $\chi_n \to T^*_\lambda\Omega$ in norm. Then by Lemma 1 of [34]

$$\langle V(\psi, f)\chi_n, \xi\rangle = 0 = \langle \chi_n, V(\psi, f)^*\xi\rangle.$$

Now let n go to infinity we have

$$\langle T^*_\lambda\Omega, V(\psi, f)^*\xi\rangle = 0 = \langle V(\psi, f)T^*_\lambda\Omega, \xi\rangle,$$

and (1) is proved.

Ad (2): Since $T^*_\lambda T^*_\mu = \sum_\nu T^*_\nu E(T_\nu T^*_\lambda T^*_\mu)$, it follows that from (1) if $E(T_\nu T^*_\lambda T^*_\mu) \neq 0$, then $E(T_\nu T^*_\lambda T^*_\mu)^* \in \mathcal{A}(I)$ is an isometry up to non-zero constant, and so $H_\nu = T^*_\nu \overline{\mathcal{A}(I)\Omega} \subset T^*_\lambda T^*_\mu \overline{\mathcal{A}(I)\Omega} \subset \overline{V(\lambda)V(\mu)H_0}$. ■

Lemma 3.4. *Suppose that $H_\lambda \in H_{\mathcal{C}}$ with $h_\lambda = 1$. Let $\psi \in H_\lambda \cap (h \ominus g)$, and $f = f^*$ a smooth function with support in I. Then* $\exp(V(\psi, f)) \in \mathcal{C}$.

Proof. Let $E_{\mathcal{C}} : \mathcal{B} \to \mathcal{C}$ be the conditional expectation which is implemented by the projection $P_{\mathcal{C}}$ on $H_{\mathcal{B}}$ with range $H_{\mathcal{C}}$. We first show that

$$P_{\mathcal{C}} V(\psi, f) \exp(tV(\psi, f))\Omega = V(\psi, f) P_{\mathcal{C}} \exp(tV(\psi, f))\Omega.$$

For any $b \in \mathcal{B}(I')$ we have

$$\begin{aligned}\langle P_{\mathcal{C}} V(\psi, f) \exp(tV(\psi, f))\Omega, b\Omega\rangle &= \langle V(\psi, f) \exp(tV(\psi, f))\Omega, E_{\mathcal{C}}(b)\Omega\rangle \\ &= -\langle \exp(tV(\psi, f))\Omega, V(\psi, f) E_{\mathcal{C}}(b)\Omega\rangle.\end{aligned}$$

Since by Prop. 2.3 of [34] $V(\psi, f)$ is skew self adjoint and is affiliated with $\mathcal{B}(I)$, and note that $V(\psi, f)\Omega \in H_\lambda \subset H_{\mathcal{C}}$, it follows that

$$\begin{aligned}\langle \exp(tV(\psi, f))\Omega, V(\psi, f) E_{\mathcal{C}}(b)\Omega\rangle &= \langle \exp(tV(\psi, f))\Omega, E_{\mathcal{C}}(b) V(\psi, f)\Omega\rangle \\ &= \langle \exp(tV(\psi, f))\Omega, P_{\mathcal{C}} b V(\psi, f)\Omega\rangle \\ &= \langle P_{\mathcal{C}} \exp(tV(\psi, f))\Omega, V(\psi, f) b\Omega\rangle.\end{aligned}$$

By (2) of Lemma 4 in [34] $P_{\mathcal{C}} \exp(tV(\psi, f))\Omega \in H(\infty)$, and $H(\infty)$ is in the domain of skew self adjoint operator $V(\psi, f)$. It follows that

$$\langle P_{\mathcal{C}} \exp(tV(\psi, f))\Omega, V(\psi, f) b\Omega\rangle = -\langle V(\psi, f) P_{\mathcal{C}} \exp(tV(\psi, f))\Omega, b\Omega\rangle,$$

and we have shown that

$$\langle P_{\mathcal{C}} V(\psi, f) \exp(tV(\psi, f))\Omega, b\Omega\rangle = \langle V(\psi, f) P_{\mathcal{C}} \exp(tV(\psi, f))\Omega, b\Omega\rangle.$$

By Reeh-Schleder Th. we have shown that

$$P_{\mathcal{C}} V(\psi, f) \exp(tV(\psi, f))\Omega = V(\psi, f) P_{\mathcal{C}} \exp(tV(\psi, f))\Omega.$$

Set $F(t) := \langle P_{\mathcal{C}} \exp(tV(\psi, f))\Omega, \exp(tV(\psi, f))\Omega\rangle$. Then $F(0) = 1$ and

$$\begin{aligned}F'(t) = \langle V(\psi, f) \exp(tV(\psi, f))\Omega, P_{\mathcal{C}} \exp(tV(\psi, f))\Omega\rangle + \\ \langle \exp(tV(\psi, f))\Omega, P_{\mathcal{C}} V(\psi, f) \exp(tV(\psi, f))\Omega\rangle = 0\end{aligned}$$

where we have used

$$P_{\mathcal{C}} V(\psi, f) \exp(tV(\psi, f))\Omega = V(\psi, f) P_{\mathcal{C}} \exp(tV(\psi, f))\Omega$$

and $P_{\mathcal{C}} \exp(tV(\psi, f))\Omega \in H(\infty)$, and $H(\infty)$ is in the domain of skew self adjoint operator $V(\psi, f)$. It follows that $F(t) = 1$ and we conclude that

$$\exp(tV(\psi, f))\Omega \in H_{\mathcal{C}}$$

which proves our lemma. ∎

The following uses an analogue of VOA statement that weight 1 space has a Lie algebra structure.

Lemma 3.5. *If $H_\lambda \subset H_{\mathcal{C}}$ with $h_\lambda = 1$, then $\mathcal{C} = \mathcal{B}$.*

Proof. By Lemma 3.4 for any $\psi \in H_\lambda \cap h$, and $f = f^*$ a smooth function with support in I, we have $\exp(V(\psi, f)) \in \mathcal{C}$. Since the conformal inclusions are maximal, it follows that the Lie algebra generated by g and ψ is in fact Lie algebra h. By Lie's formula, if $\exp(iV(\psi_j, f_j)) \in \mathcal{C}(I), j = 1, 2$ then

$$((\exp(V(\psi_1, f_1)/n) \exp(V(\psi_2, f_2)/n)(\exp(-V(\psi_1, f_1)/n) \exp(-V(\psi_2, f_2)/n))^{n^2}$$

converges strongly to

$$\exp([V(\psi_1, f_1), V(\psi_1, f_1)]).$$

On the other hand

$$[V(\psi_1, f_1), V(\psi_2, f_1)] = V([\psi_1, \psi_2], fg) + \langle \psi_1, \psi_2 \rangle \int_{S^1} f_1 f_2 dz/z.$$

It follows that for any $\psi \in h$, and smooth functions $f = f^*$ with support in I we have that

$$\exp(V(\psi, f)) \in \mathcal{C}(I).$$

Since $\mathcal{B}(I)$ is generated as a von Neumann algebra by such elements, we have shown that $\mathcal{C} = \mathcal{B}$. ■

3.2 Induction of the adjoint representation

The following is a key observation in this section, and is already implicitly contained in (3) of Lemma 2.33 in [39].

Proposition 3.6. *Suppose that $\mathcal{C}$ contains no weight 1 element except those in $\mathcal{A}$, then $a^{\mathcal{C}}_{v_0}$ is irreducible.*

Proof. By Lemma 2.13 we have

$$[{v_0}^2] = [1]+2[v_0]+[(2,0,...,0,2)]+[(0,1,0,...,1,0)]+[(0,1,0,...,0,2)]+[(2,0,...,0,1,0)]$$

By computing the conformal dimensions of the descendants of ${v_0}^2$ using equation (5) we have

$$h_{(2,0,...,0,2)} = \frac{2+2n}{k+n}, h_{(0,1,...,0,2)} = h_{(2,0,...,1,0)} = \frac{2n}{k+n}, h_{(0,1,...,1,0)} = \frac{2n-2}{k+n}$$

Hence if $\mathcal{C}$ contains no weight 1 element except those in $\mathcal{A}$, then

$$\langle a^{\mathcal{C}}_{v_0}, a^{\mathcal{C}}_{v_0} \rangle = \langle H_{\mathcal{C}}, v_0 v_0 \rangle = 1$$

where recall that we use $H_{\mathcal{C}}$ to denote the restriction of the vacuum representation of $\mathcal{C}$ to $\mathcal{A}$, and the proposition is proved. ■

Lemma 3.7. *Suppose that $\epsilon(\lambda, v_0)\epsilon(v_0, \lambda) = 1$, then $\lambda = \omega^i$ for some $0 \leq i \leq n$.*

Proof. By definition we have

$$\frac{S_{v_0\lambda}}{S_{1\lambda}} = d_{v_0}.$$

From $[v][\bar{v}] = [1] + [v_0]$ we have

$$\frac{S_{v\lambda}}{S_{1\lambda}}\frac{S_{\bar{v}\lambda}}{S_{1\lambda}} = d_{v_0} + 1 \leq d_v d_{\bar{v}} = d_{v_0} + 1$$

It follows that we must have $|\frac{S_{v\lambda}}{S_{1\lambda}}| = d_v$. For any positive integer k, suppose that $[v^k] = \sum_\mu m_\mu[\mu]$, then we have

$$d_v^n = |\sum_\mu m_\mu \frac{S_{\mu\lambda}}{S_{1\lambda}}| \leq \sum_\mu m_\mu |\frac{S_{\mu\lambda}}{S_{1\lambda}}| \leq \sum_\mu m_\mu d_\mu = d_v^n.$$

It follows that $|\frac{S_{\mu\lambda}}{S_{1\lambda}}| = d_\mu, \forall \mu \prec v^k$. Since every irrep of $\mathcal{A}$ occurs in some v^k, it follows that we must have $|\frac{S_{\mu\lambda}}{S_{1\lambda}}| = d_\mu, \forall \mu$. Square both sides and sum over μ, we have proved that $d_\lambda = 1$, and hence the Lemma. ■

3.3 List of intermediate subnets from conformal inclusions

Theorem 3.8. *(1) For the subnet $\mathcal{A} \subset \mathcal{B}$ corresponding to conformal inclusions in 8, when n is odd (resp. even) the intermediate subnet $\mathcal{C}$ are in one to one correspondence with the abelian subgroup $\mathbb{Z}_n$ (resp. $\mathbb{Z}_{n/2}$) generated by ω, (resp. ω^2) i.e., if $\omega^i, ik = n$ (resp. $\omega^{2i}, 2ik = n$) is a generator of this subgroup, then the spectrum of $\mathcal{C}$ is $H_\mathcal{C} = \sum_{1\leq j\leq k} H_{\omega^{ij}}$ (resp. $H_\mathcal{C} = \sum_{1\leq j\leq k} H_{\omega^{2ij}}$);*

(2): For the subnet $\mathcal{A} \subset \mathcal{B}$ corresponding to conformal inclusions in 6, 7, when n is odd there is no intermediate subnet. When $n = 2m$ is even, the only nontrivial intermediate subnet $\mathcal{C}$ is a $\mathbb{Z}_2$ extension of $\mathcal{A}$ by the simple current ω^m, i.e., the spectrum is $H_\mathcal{C} = H_0 + H_{\omega^m}$;

Proof. Ad (1): By Lemma 3.5 we can assume that $\mathcal{C}$ has no weight 1 elements besides those of $\mathcal{A}$. By Prop. 3.6 we know that $a_{v_0}^\mathcal{C}$ is irreducible. In the case of conformal inclusions in 8, since the vector representation of LH, when restricting to $\mathcal{A}$, contains the adjoint representation, it follows from (4) of 2.6 that $a_{v_0}^\mathcal{C}$ must contain a DHR representation of $\mathcal{C}$. Since $a_{v_0}^\mathcal{C}$ is irreducible, it follows that $a_{v_0}^\mathcal{C}$ is a DHR representation of $\mathcal{C}$, i.e., $[a_{v_0}^\mathcal{C}] = [\tilde{a}_{v_0}^\mathcal{C}]$. By Lemma 2.8 we must have for any $\lambda \in H_\mathcal{C}$, $\epsilon(\lambda, v_0)\epsilon(v_0, \lambda) = 1$. By Lemma 3.7 we conclude that $\lambda = \omega^i$ for some $0 \leq i \leq n$. (1) now follows easily by inspection of the spectrum of $\mathcal{A} \subset \mathcal{B}$ in [1].

Ad (2): In the case of conformal inclusions in 6 (resp. 7) , we note that the vector representation of LH, when restricting to $\mathcal{A}$, contains the antisymmetric representation $(0, 1, 0, ..., 0)$ (resp. symmetric representation $(2, 0, 0, ...0)$ of $\mathcal{A}$).) Since

$$\langle a_{v^2}^\mathcal{C}, a_{v^2}^\mathcal{C}\rangle = \langle [a_{(2,0,...,0)}^\mathcal{C}] + [a_{(0,1,0,...,0)}^\mathcal{C}], [a_{(0,2,0,...,0)}^\mathcal{C}] + [a_{(1,1,0,...,0)}^\mathcal{C}]\rangle = \langle a_{v\bar{v}}^\mathcal{C}, a_{v\bar{v}}^\mathcal{C}\rangle = 2,$$

it follows that both $a^{\mathcal{C}}_{(2,0,...,0)}$ and $a^{\mathcal{C}}_{(0,1,0,...,0)}$ are irreducible. Hence as in the proof of (1), for the case of conformal inclusions in 6 (resp. 7), $a^{\mathcal{C}}_{(0,1,0,...,0)}$ (resp. $a^{\mathcal{C}}_{(2,0,...,0)}$) are DHR representations of $\mathcal{C}$. It follows that if $\lambda \in H_{\mathcal{C}}$, then $\epsilon(\lambda,(0,1,0,...,0))\epsilon((0,1,0,...,0),\lambda) = 1$ (resp.$\epsilon(\lambda,(2,0,...,0))\epsilon((2,0,...,0),\lambda) = 1$). Similarly $\epsilon(\lambda,(0,0,...,0,1,0))\epsilon((0,0,...,1,0),\lambda) = 1$ (resp.$\epsilon(\lambda,(0,0,...,2))\epsilon((0,0,...,2),\lambda) = 1$).

Since by Lemma 2.14 v_0 appears in the product of $(0,1,0,...,0)$ (resp. $(2,0,...,0)$)and its conjugate, it follows that $\epsilon(\lambda,v_0)\epsilon(v_0,\lambda) = 1$. By Lemma 3.7 we conclude that $\lambda = \omega^i, 1 \leq i \leq n$, and (2) follows by inspection of the spectrum of $\mathcal{A} \subset \mathcal{B}$ as given in [21]. ∎

We note that the same idea in the proof of Theorem above gives a proof of the following:

Corollary 3.9. *Suppose $\mathcal{A}_{SU(n)_k} \subset \mathcal{C}, n \neq n, n \pm 2$, and there is a representation of $\mathcal{C}$, when restricting to $\mathcal{A}_{SU(n)_k}$, contains v_0. Then $\mathcal{C}$ is an extension by simple currents.*

Remark 3.10. *Since conformal inclusion $SU(2)_{10} \subset Spin(5)_1$ is not a simple current extension, this example shows that the condition in the above corollary is necessary. In fact in this case the adjoint representation of $\mathcal{A}_{SU(2)_{10}}$ does not appear in the restriction of any irreps of $\mathcal{A}_{Spin(5)_1}$.*

Theorem 3.11. *For the subnet $\mathcal{A} \subset \mathcal{B}$ associated with 9, let $(n,m) = p, n = n_1 p, m = m_1 p$. Then the intermediate subnets $\mathcal{C}$ are in one to one correspondence with the subgroup of $\mathbb{Z}_p$, i.e.,each such $\mathcal{C}$ has spectrum $H_{\mathcal{C}} = \sum_{0 \leq l \leq k_2} H_{(\omega^{n_1 k_1 l}, \dot{\omega}^{m_1 k_1 l})}$ with $k_1 k_2 = p$, where we use $\dot{\lambda}$ to denote the highest weights of $SU(m)_n$.*

Proof. Since $\mathcal{A}_{SU(n)_m} \subset \mathcal{B}$ is normal (cf. §4 of [35]), it follows that for each $(\lambda, \dot{\lambda}) \in H_{\mathcal{C}}$, we must have $[a^{\mathcal{C}}_\lambda] = [a^{\mathcal{C}}_{\dot{\lambda}}]$, and $\lambda \to a^{\mathcal{C}}_\lambda$ is a ring isomorphism. So if $(\lambda_i, \dot{\lambda}_i) \in H_{\mathcal{C}}, i = 1,2$, and

$\lambda_3 \prec \lambda_1\lambda_2$, then $[a^{\mathcal{C}}_{\lambda_3}] \prec [a^{\mathcal{C}}_{\lambda_1\lambda_2}] = [a^{\mathcal{C}}_{\dot{\lambda}_1\dot{\lambda}_2}]$, it follows there must be a $\dot{\lambda}_3$ such that $[a^{\mathcal{C}}_{\lambda_3}] = [a^{\mathcal{C}}_{\dot{\lambda}_3}]$ and $(\lambda_3, \dot{\lambda}_3) \in H_{\mathcal{C}}$.

It follows that $\mathcal{C}$ are in one to one correspondence with the set $R_{\mathcal{C}}$ of λ with color zero mod n which are closed under conjugation and fusion product. If $R_{\mathcal{C}}$ contains any λ with $d_\lambda \neq 1$, by Lemma 2.14 we have $v_0 \in R$, and it follows that R contains all λ with color zero mod n, in which case $\mathcal{C} = \mathcal{B}$. Now assume that $d_\lambda = 1$ if $\lambda \in R_{\mathcal{C}}$. The $R_{\mathcal{C}}$ must be a subgroup generated by $\omega^{n_1 k_1}, k_1 k_2 = p$. Since the color of elements in $R_{\mathcal{C}}$ is zero mod n, our Theorem follows. ∎

By checking the list of intermediate subnets from Th. 3.8 and Th. 3.11 we immediately have:

Corollary 3.12. *Conjecture 1.2 is true for $\mathcal{A}_{G_K} \subset \mathcal{A}_{H_1}$ where $G_K \subset H_1$ are conformal inclusions in 6,7,8 and 9.*

For the conformal inclusions $G_k \subset H_1$, we write $V_{\mathcal{A}}$ (resp. $V_{\mathcal{B}}$) the VOA (cf. [9]) associate with affine $\hat{g}$ at level k (resp. affine $\hat{h}$ at level 1) We have natural inclusion $V_{\mathcal{A}} \subset V_{\mathcal{B}}$. We are interested in VOA $V_{\mathcal{C}}$ such that $V_{\mathcal{A}} \subset V_{\mathcal{C}} \subset V_{\mathcal{B}}$. We say a VOA is

simple it is irreducible as a representation over itself. Note that $V_{\mathcal{C}}$ will be direct sum of $\hat{g}$ modules, and we can write $V_{\mathcal{C}} = \oplus H_\lambda$ and we refer to those λ which appear in $V_{\mathcal{C}}$ as the *spectrum* of $V_{\mathcal{C}}$.

Proposition 3.13. *For any simple VOA $V_{\mathcal{C}}$ such that $V_{\mathcal{A}} \subset V_{\mathcal{C}} \subset V_{\mathcal{B}}$, there corresponds a unique intermediate subnet $\mathcal{C}$ such the spectrum of $\mathcal{A} \subset \mathcal{C}$ is the same as the spectrum of $V_{\mathcal{A}} \subset V_{\mathcal{C}}$.*

Proof. Fix an interval I. For each λ in the spectrum of $V_{\mathcal{A}} \subset V_{\mathcal{C}}$, denote by T_λ be as in Lemma 2.3. By Lemma 3.3, it follows that if $E(T_\nu T_\lambda^* T_\mu^*) \neq 0$ where λ, μ are in the spectrum of $V_{\mathcal{A}} \subset V_{\mathcal{C}}$, then ν is also in the spectrum of $V_{\mathcal{A}} \subset V_{\mathcal{C}}$. If λ is in the spectrum of $V_{\mathcal{A}} \subset V_{\mathcal{C}}$ but $\bar{\lambda}$ is not, then by the remark after Definition 3.2 the action of $V_{\mathcal{C}}$ on H_λ will span an invariant subspace of $H_{\mathcal{C}}$ which does not contain H_0, contradicting our assumption that $\mathcal{C}$ is simple. By (2) of Lemma 2.3, $\mathcal{A}, T_\lambda$ where λ is in the spectrum of $V_{\mathcal{A}} \subset V_{\mathcal{C}}$ generate an intermediate subnet $\mathcal{C}$ with its spectrum the same as the spectrum of $V_{\mathcal{A}} \subset V_{\mathcal{C}}$. ∎

The above proposition immediately implies the following theorem:

Theorem 3.14. *The set of intermediate simple VOA $V_{\mathcal{C}}$ in $V_{\mathcal{A}} \subset V_{\mathcal{B}}$ for conformal inclusions 6,7,8 and 9 are in one to one correspondence with the set of intermediate subnets $\mathcal{C}$ of $\mathcal{A} \subset \mathcal{B}$ with the same spectrum as given in Th. 3.8 and Th. 3.11.*

Remark 3.15. *We note that the simple intermediate VOAs in the above theorem are simple current extensions of affine VOAs, and they are well understood in VOA literature (cf.[6]).*

4 Verifying Conjecture 1.1 for Jones-Wassermann subfactors

In this section we extend the results in Cor. 5.23 of [39].

Let λ be an irreducible representation of $\mathcal{A}_{SU(n)_k}$ localized on $I, M := \mathcal{A}_{SU(n)_k}(I)$. Suppose $\lambda = c_1 c_2$ where $c_i \in \mathrm{End}(M), i = 1, 2, c_1(M)$ is an intermediate subfactor of $\lambda(M) \subset M$. We note that $c_1 \bar{c}_1 \prec \lambda\bar{\lambda}$. We say the intermediate subfactor $c_1(M)$ is of *abelian type* if $[c_1\bar{c}_1] = \sum_{1 \leq i \leq j} [\omega^{ij_1}], jj_1 = n$. The following Lemma appears as Lemma 5.22 in the correction of proof of [39] and we include its proof:

Lemma 4.1. *Assume that $Z^{c_1}_{1\mu} = \delta_{1\mu}, \forall \mu$ where Z^{c_1} is defined as in Definition 2.9. Then $\langle c_1 c_2, c_1 c_2 \rangle = \langle c_1 \bar{c}_1, \bar{c}_2 c_2 \rangle$.*

Proof By §2 of [10] we have $Z^{c_1}_{\mu_1 \mu_2} = \delta_{\mu_1 \tau(\mu_2)}$ where $\mu \to \tau(\mu)$ is an order two automorphism of fusion algebra. It follows that $[\tilde{a}_\mu] = [a_\tau(\mu)]$, and by [3] irreducible sectors of $\bar{c}_1 \nu c_1$ are of the form $a_\mu, \forall \mu$. Since

$$\langle c_2 \bar{c}_2, a_\mu \rangle = \langle c_2, a_\mu c_2 \rangle = \langle c_2, c_2 \mu \rangle = \langle \bar{c}_2 c_2, a_\mu \rangle = \langle a_{\bar{c}_2 c_2}, a_\mu \rangle,$$

we conclude that $[c_2\bar{c}_2] = [a_{\bar{c}_2 c_2}]$, and

$$\langle c_1\bar{c}_1, \bar{c}_2 c_2\rangle = \langle c_1, \bar{c}_2 c_2 c_1\rangle = \langle c_1, c_1 a_{\bar{c}_2 c_2}\rangle = \langle c_1, c_1 c_2\bar{c}_2\rangle = \langle c_1 c_2, c_1 c_2\rangle$$

■

Theorem 4.2. *(1) Suppose that $k \neq n-2, n+2, n$. then λ is maximal iff there is no $1 \leq i \leq n-1$ such that $[\omega^i\lambda] = [\lambda]$;*

(2) When λ is not maximal, the maximal intermediate subfactor is either abelian type or at most one given by $c_1(M)$ with $\lambda = c_1 c_2, [\bar{c}_2][c_2] = [1] + [\omega^m], n = 2m$.

Proof Ad (1): (1) is Cor. 5.23 in [39]. We include its proof which will be modified in our proof of (2).

When $k = 1$ the Cor. is obvious. By Lemma 2.33 of [39] we can assume that $k \geq 2$ and $d_{v_0} > 1$. As in the proof of Cor. 5.21 in [39], λ is maximal implies that there is no $1 \leq i \leq n-1$ such that $[\omega^i\lambda] = [\lambda]$. Now suppose that there is no $1 \leq i \leq n-1$ such that $[\omega^i\lambda] = [\lambda]$. If $S_{v\lambda} \neq 0$, then λ is maximal by Cor. 5.20 of [39]. If $k = 2$, the S matrix elements are equal to that of S matrix elements for $SU(2)_n$ up to phase factors, and it follows easily that $S_{v\lambda} \neq 0$ if there is no $1 \leq i \leq n-1$ such that $[\omega^i\lambda] = [\lambda]$.

Suppose that $k \geq 3, S_{v\lambda} = 0$. Since $[v\bar{v}] = [1] + [v_0]$ we have $S_{v_0\lambda} = -S_{1\lambda} \neq 0$. Assume that M_1 is an intermediate subfactor between $\lambda(M)$ and M, and $\lambda = c_1 c_2$ with $c_1(M) = M_1$ and $c_1 = c_1' c_1''$ as in Prop. 2.7. Apply Lemma 2.20 of [39] we have $\langle a_{v_0}^{c_1'}, \tilde{a}_{v_0}^{c_1'}\rangle \geq 1$. By Lemma 2.33 of [39] we must have $[a_{v_0}^{c_1'}] = [\tilde{a}_{v_0}^{c_1'}]$ and by Lemma 2.36 of [39] $[c_1'\bar{c}_1'] = \sum_{1\leq j\leq n/j_1}[\omega^{jj_1}]$. By Frobenius reciprocity we have $[\omega^{j_1}c_1'] = [c_1'.]$ Since $\lambda = c_1' c_1'' c_2$, $[\omega^{j_1}\lambda] = [\lambda]$, and by assumption $j_1 = n$ and $[c_1'\bar{c}_1'] = [1]$. By Prop. 2.7 we must have $Z_{\mu 1}^{c_1} = \delta_{\mu 1}, \forall \mu$. By §2 of [10] we have $Z_{\mu_1\mu_2}^{c_1} = \delta_{\mu_1 \tau(\mu_2)}$ where $\tau(\mu) = \omega^{m\mathrm{col}(\mu)}\mu$ or $\tau(\mu) = \omega^{m\mathrm{col}(\mu)}\bar{\mu}, m \geq 0$. We claim that in fact $[\omega^m] = [1]$ and $\tau(\mu) = \mu$. First we show that $\tau(\mu) = \omega^{m\mathrm{col}(\mu)}\mu$. If instead $\tau(\mu) = \omega^{m\mathrm{col}(\mu)}\bar{\mu}$, since $k \geq 3$, $\tau((0,1,0,...,0)) \neq (0,1,0,...,0)$, by Lemma 2.20 of [39] we must have $S_{\lambda(0,1,0,...,0)} = 0$. From the fusion rule

$$[(0,1,0,...,0)(0,0,...,0,2)] = [(0,1,0,...,0,2)] + [v_0]$$

we must have $S_{\lambda(0,1,0,...,0,2)} \neq 0$. By Lemma 2.20 of [39] we must have $\tau((0,1,0,...,0,2)) = (0,1,0,...,0,2) = (2,0,0,...,1,0)$, a contradiction. So we conclude that $\tau(\mu) = \omega^{m\mathrm{col}(\mu)}\mu, \forall \mu$. It follows that $[\tilde{a}_\mu] = [a_{\omega^{m\mathrm{col}(\mu)}} a_\mu]$, and in particular $[\tilde{a}_v] = [a_{\omega^m} a_v]$. So we have

$$[\omega^m v c_1] = [c_1\tilde{a}_v] = [c_1 a_v] = [v c_1],$$

and similarly $[c_2\omega^{-m}\bar{v}] = [c_2\bar{v}]$. If $[\omega^m] \neq [1]$, by our assumption on λ we have $\omega^m \not\prec c_1\bar{c}_1, \omega^m \not\prec \bar{c}_2 c_2$. On the other hand we have

$$\langle \bar{v}\omega^m v, c_1\bar{c}_1\rangle \geq 1, \langle \bar{v}\omega^m v, \bar{c}_2 c_2\rangle \geq 1$$

It follows that $\omega^m v_0 \prec c_1\bar{c}_1, \omega^m v_0 \prec \bar{c}_2 c_2$, and $\langle c_1\bar{c}_1, \bar{c}_2 c_2\rangle \geq 2$. By Lemma 4.1 we conclude that $\lambda = c_1 c_2$ is not irreducible, contradicting our assumption. Hence $[\omega^m] =$

[1] and $Z_{\mu_1\mu_2} = \delta_{\mu_1\mu_2}$. The rest of the proof now follows in exactly the same way as in the proof of Prop. 5.20 of [39].

Ad (2): As in the proof of (1) we assume that $\lambda = c_1c_2$ with $c_1(M)$ a nontrivial maximal intermediate subfactor, and $c_1 = c_1'c_1''$. By our assumption we must have $c_1 = c_1'$ if $[c_1'] \neq [1]$. In this case as in the proof of (1) above we must have $[c_1\bar{c}_1] = \sum_{1\leq j\leq n/j_1}[\omega^{jj_1}]$.

Now suppose that $[c_1'] = [1]$. Then as above we have $Z^{c_1}_{\mu\gamma} = \delta_{\mu,\omega^{\mu\mathrm{col}(\gamma)}\gamma}$. By Corollary 3.14 of [39] we can find $c \prec \mu c_1$ for some μ such that

$$[c\bar{c}] = \sum_{1\leq i\leq p} \omega^{li}, pl = n.$$

Since $Z^c = Z^{c_1}$, it follows the left local support of c is trivial.

If $[\omega^m] = [1]$, then we are as in the end of proof of (1), and in that case $[c_1] = [1]$, contradicting our assumption that $c_1(M)$ a nontrivial maximal intermediate subfactor. So $[\omega^m] \neq [1]$. If $\omega^m \prec c_1\bar{c}_1$, then by maximality of $c_(M)$ we have $[c_1\bar{c}_1] = \sum_i[\omega^{qi}]$, i.e., $c_1(M)$ comes from abelian part of λ. Now assume that $\omega^m \not\prec c_1\bar{c}_1$, then as in (1) we must have $\omega^m \prec \bar{c}_2c_2$.

Since $c \prec \mu c_1$ for some μ, we have $c\bar{c} \prec \mu c_1\bar{c}_1\bar{\mu} \prec \mu\lambda\bar{\lambda}\bar{\mu}$, so $\mathrm{col}(\omega^{li}) = 0 \bmod n$, so we have $n|li$.

Similarly since $\omega^m \prec \bar{c}_2c_2$, $\mathrm{col}(\omega^m) = 0 \bmod n$. On the other hand since the map $\mu \to \omega^{m\mathrm{col}(\mu)}\mu$ has order two, it follows that $\omega^{2m} = 1$. So we must have $n = 2m$.

From

$$h_{\omega^{li}} = \frac{kli}{n}\frac{n-li}{2},$$

it follows that $h_{\omega^{li}} \in \mathbb{Z}$ if i is even. These ω^{li} with i even will generate local simple currents (cf Definition 2.3 and Prop. 2.15 of [36]), and it follows that the left local support of c is nontrivial if $li \neq 0 \mathrm{mod} n$ for some even i. So we conclude that $2l = 0 \mathrm{mod} n$, and $[c\bar{c}] = [1] + [\omega^m]$.

Note that there are λ_1, λ_2 such that $c_1 \prec \lambda_1 c, \bar{c}_2 \in \lambda_2 c$. From $\langle c_1, \lambda_1 c\rangle = \langle c_1\bar{c}, \lambda_1\rangle \geq 1$, we have $d_{c_1}\sqrt{2} \geq d_{\lambda_1}$, and similarly $d_{\bar{c}_2}\sqrt{2} \geq d_{\lambda_2}$.

Since $[\omega^m c_1] \neq [c_1]$, we have $[\lambda_1 c] \succ [c_1] + [\omega^m c_1]$, and by computing statistical dimension $[\lambda_1 c] = [c_1] + [\omega^m c_1]$, and $[c_1\bar{c}] = [\lambda_1]$.

Similarly if $\lambda_2 c$ is not irreducible, we must have $[\bar{c}_2\bar{c}] = [\lambda_2]$. But we have

$$\langle \bar{c}_2\bar{c}, \bar{c}_2\bar{c}\rangle = \langle \bar{c}_2\bar{c}cc_2, 1\rangle = \langle \bar{c}_2 a_{c\bar{c}} c_2, 1\rangle = \langle c\bar{c}\bar{c}_2c_2, 1\rangle \geq 2,$$

where we have used $[\bar{c}c] = [a_{c\bar{c}}]$ and $\bar{c}_2c_2 \succ [1] + [\omega^m]$. From this we conclude that $[\bar{c}_2] = [\lambda_2 c]$.

We have $[\lambda] = [c_1c_2] = [c_1\bar{c}\bar{\lambda}_2] = [\lambda_1\bar{\lambda}_2]$. By Lemma 2.14 , we must have $\bar{\lambda}_2 = \omega^i, [\lambda] = [c_1\bar{c}\omega^i]$.

Now we check that the intermediate subfactor $c_1(M)$ is uniquely fixed. Suppose there is an intermediate subfactor $f_1(M)$ such that $[\lambda] = [f_1f_2]$ and $[\bar{f}_2f_2] = [1] + [\omega^m]$. Then $\bar{c}\bar{f}_2$ has statistical dimension two and decompose into sum of two irreducible endomorphisms, it follows that there is an automorphism α such that $[f_2] = [\alpha\bar{c}\omega^i]$, and

$[f_1\alpha] = [c_1\beta]$ for some automorphism $\beta \prec \bar{c}c$. By Cor. 2.4 of of [37] the intermediate subfactor $f_1(M)$ is determined by equivalence class $[f_1, f_2]$ with equivalence relation $[f_1, f_2] \sim [f_1\rho, \rho^{-1}f_2]$ where ρ is any automorphism. We have

$$[f_1, f_2] = [c_1\beta\alpha^{-1}, \alpha\bar{c}\omega^i] \sim [c_1\beta, \bar{c}\omega^i] \sim [c_1, \beta^{-1}\bar{c}\omega^i] = [c_1, \bar{c}\omega^i].$$

Corollary 4.3. *Suppose that $k \neq n-2, n+2, n$. Then each irreducible representation λ of $\mathcal{A}_{SU(n)_k}$ verifies both maximal and minimal version of Conjecture 1.1.*

Proof. Since the dual of λ is $\bar{\lambda}$, it is sufficient to verify the maximal version of Conjecture 1.1. We may assume that $d_\lambda > 1$. By Lemma 2.14

$$[\lambda][\bar{\lambda}] = \sum_{1\leq i\leq p} [\omega^{iq}] + [v_0] + ...$$

where $pq = n$, and ... are possible additional irreps. We note that the set of maximal intermediate subfactors for λ coming the abelian part are bounded by $p-1$, and by Th. 4.2, there is at most one more maximal intermediate subfactor, and our corollary follows. ∎

Remark 4.4. *It will be interesting to remove the condition $k \neq n, n\pm 2$ in Th.4.2 and Cor. 4.3. This condition is used in the proof of Th. 4.2 to ensure that $a_{v_0}^{c_1}$ is irreducible. One can remove this condition if one can find a different way of proving that $a_{v_0}^{c_1}$ is irreducible.*

References

[1] D. Altschüler, M. Bauer and C. Itzykson, *The branching rules of conformal embeddings*, Comm. Math. Phys., **132** (1990), 349-364.

[2] J. Böckenhauer, D. E. Evans, Modular invariants, graphs and α-induction for nets of subfactors. I., *Comm.Math.Phys.,* **197**, 361-386, 1998.

[3] J. Böckenhauer, D. E. Evans, Y. Kawahigashi, Chiral structure of modular invariants for subfactors, *Comm.Math.Phys.,* **210**, 733-784, 2000.

[4] D. Bisch and V. F. R. Jones *Algebras associated to intermediate subfactors,* Invent. Math. 128 (1997), no. 1, 89–157.

[5] C. Dong, *Introduction to vertex operator algebras I,* *Sūrikaisekikenkyūsho Kōkyūroku*, No. 904 (1995), 1-25. Also see q-alg/9504017.

[6] C. Dong and J. Lepowsky, *Generalized vertex algebras and relative vertex operators,* Progress in Mathematics, 112 (1993).

[7] J. Fröhlich and F. Gabbiani, *Operator algebras and Conformal field theory,* Comm. Math. Phys., **155**, 569-640 (1993).

[8] I. B. Frenkel, J. Lepowsky and J. Ries, *Vertex operator algebras and the Monster*, Academic, New York, 1988.

[9] I. Frenkel and Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras,* Duke Math. Journal (1992), Vol. 66, No. 1, 123-168.

[10] T. Gannon, P. Ruelle, and M. A. Walton, *Automorphism modular invariants of current algebras,* Comm. Math. Phys. 179 (1996), no. 1, 121–156.

[11] F. Goodman and H. Wenzl, *Littlewood-Richardson coefficients for Hecke algebras at roots of unity,* Adv. Math. 82 (1990), no. 2, 244–265.

[12] P. Grossman and Vaughan F. R. Jones, *Intermediate subfactors with no extra structure,* J. Amer. Math. Soc. 20 (2007), no. 1, 219–265.

[13] D. Guido & R. Longo, *The conformal spin and statistics theorem*, Commun. Math. Phys. **181** (1996) 11–35.

[14] R. Guralnick and F. Xu, *On a subfactor generalization of Wall's conjecture.* J. Algebra 332 (2011), 457468

[15] V. F. R. Jones, *Fusion en algbres de von Neumann et groupes de lacets (d'aprs A. Wassermann). (French) [Fusion in von Neumann algebras and loop groups (after A. Wassermann)],* Sminaire Bourbaki, Vol. 1994/95. Astrisque No. 237 (1996), Exp. No. 800, 5, 251–273.

[16] V. G. Kac, *Vertex algebras for beginners*, AMS, 1997.

[17] V. G. Kac, "Infinite Dimensional Lie Algebras", 3rd Edition, Cambridge University Press, 1990.

[18] V. G. Kac, R. Longo and F. Xu, *Solitons in affine and permutation orbifolds,* Comm. Math. Phys. 253 (2005), no. 3, 723–764.

[19] A. Tsuchiya and Y. Kanie, *Vertex Operators in conformal field theory on P^1 and monodromy representations of braid group,* Adv. Studies in Pure Math. 16 (88), 297-372.

[20] V. G. Kac and M. Wakimoto, *Modular and conformal invariance constraints in representation theory of affine algebras*, Advances in Math., **70**, 156-234 (1988).

[21] F. Levstein and J. I. Liberati, *Branching rules for conformal embeddings,* Commun. Math. Phys. **173** (1995), 1–16.

[22] R. Longo, *Conformal subnets and intermediate subfactors*, Commun. Math. Phys. **237** n. 1-2 (2003), 7–30.

[23] R. Longo & K.-H. Rehren, *Nets of subfactors*, Rev. Math. Phys. **7** (1995) 567–597.

[24] P. Goddard, W. Nahm and D. Olive, *Symmetric spaces, Sugawara's energy momentum tensor in two dimensions and free fermions*, Phys. Lett. B 160 (1985), no. 1-3, 111–116

[25] M. Izumi, R. Longo & S. Popa, *A Galois correspondence for compact groups of automorphisms of von Neumann Algebras with a generalization to Kac algebras*, J. Funct. Analysis, **155**, 25-63 (1998).

[26] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983) 1–25.

[27] Vaughan F. R. Jones and F. Xu, *Intersections of finite families of finite index subfactors,* Internat. J. Math. 15 (2004), no. 7, 717–733.

[28] M. W. Liebeck, L. Pyber and A. Shalev, *On a conjecture of G. E. Wall,* J. Algebra 317 (2007), no. 1, 184–197.

[29] M. Pimsner, & S. Popa, *Entropy and index for subfactors*, Ann. Scient. Ec. Norm. Sup. **19** (1986), 57–106.

[30] A. Pressley and G. Segal, "Loop Groups" Oxford University Press 1986.

[31] G. E. Wall, *Some applications of the Eulerian functions of a finite group,* J. Austral. Math. Soc. 2 1961/1962 35–59.

[32] A. Wassermann, Operator algebras and Conformal field theories III, *Invent. Math.* **133** (1998), 467-538.

[33] F. Xu, *New braided endomorphisms from conformal inclusions*, Commun. Math. Phys. **192** (1998) 347–403.

[34] F. Xu, *Algebraic coset conformal field theories.* Comm. Math. Phys. 211 (2000), no. 1, 1-43.

[35] F. Xu, *Mirror extensions of local nets.* Comm. Math. Phys. 270 (2007), no. 3, 835-847.

[36] F. Xu, *An application of mirror extensions.* Comm. Math. Phys. 290 (2009), no. 1, 83-103.

[37] F. Xu, *On intermediate subfactors of Goodman-de la Harpe-Jones subfactors*, Commun. Math. Phys. 298 (2010), no.3, 707–739.

[38] F. Xu, *Some computations in the cyclic permutations of completely rational nets,* Comm. Math. Phys. 267 (2006), no. 3, 757-782.

[39] F. Xu, *On representing some lattices as lattices of intermediate subfactors of finite index* , Adv. Math. 220 (2009), no. 5, 1317-1356. Corrections in the proof of Cor. 5.23 in arXiv:math/0703248.